Discontinuous Centrally Symmetric Motions of Ultra-Relativistic Gases in the General Theory of Relativity

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The problem is considered in the self-similar formulation. Shock wave conditions are investigated. For the case in which the moving gas is connected with the stationary one via a shock wave, asymptotic solutions are given in the neighborhood of $t = 0$, $r = \infty$; $t = \infty$, and $r = 0$. A solution is given for the case in which the gas moves toward a central point, and a schematic description is given of the formulation of a central nebula.

1 IF the gravitational field due to an ideal fluid is centrally symmetric, the equations of Einstein

$$R^{i\kappa} - \tfrac{1}{2}g^{ik}R = -\varkappa[(\epsilon + p)u^i u^k - g^{ik}p] \qquad (1.1)$$

are applicable to the case of an ultra-relativistic gas with the equations of state

$$\epsilon = 3p \qquad (1.2)$$

in the reference frame, in which

$$ds^2 = Y(t, r; \sigma_1, \sigma_2, \ldots)dt^2 - x(t, r; \sigma_1, \sigma_2, \ldots)dr^2 - r^2(d\theta^2 + \sin^2\theta d\Phi^2) \qquad (1.3)$$

may be written in the form[1]

$$\left(\frac{Y'}{\sqrt{xY}}\right)' - \left(\frac{\dot{x}}{\sqrt{xY}}\right)' = \frac{2}{r^2}\sqrt{\frac{Y}{x}}\,(1 - x) + \frac{4}{r}\left(\sqrt{\frac{Y}{x}}\right)' \qquad (1.4)$$

$$4(x - 1) + 2r\left(\frac{x'}{x} - \frac{Y'}{Y}\right) = r\sqrt{\left(\frac{x'}{x} + \frac{Y'}{Y}\right)^2 - 4\frac{\dot{x}^2}{xY}} \geqslant 0 \qquad (1.5)$$

$$v = \frac{dr}{dt} = \frac{Y}{2\dot{x}}\left[\frac{4}{r}(x - 1) + \frac{x'}{x} - 3\frac{Y'}{Y}\right] \qquad (1.6)$$

$$\varkappa p = \frac{1}{2rx}\left[\frac{2}{r}(x - 1) + \frac{x'}{x} - \frac{Y'}{Y}\right] \qquad (1.7)$$

constant σ with dimensionality $T^{-2}L^{-m}$. Then from the general theory[2] it follows that the motion in this case will be self-similar.

Let us assume that

$$Y = \frac{r^2}{t^2}\,y(\xi) \qquad x = x(\xi) \qquad \xi = \sigma t^2 r^m \qquad (1.8)$$

where y is a nondimensional function. Then Eqs. (1.4) and (1.5) will take on the form

$$\xi\frac{d}{d\xi}\left[\frac{\xi}{\sqrt{xy}}\left(m^2\frac{dy}{d\xi} - 4\frac{dx}{d\xi}\right) + 6m\sqrt{\frac{y}{x}}\right] = 2\sqrt{\frac{y}{x}}(x - 3) \qquad (1.9)$$

$$4x - 8 + 2m\xi\left(\frac{1}{x}\frac{dx}{d\xi} - \frac{1}{y}\frac{dy}{d\xi}\right) = \sqrt{\left[m\xi\left(\frac{1}{x}\frac{dx}{d\xi} + \frac{1}{y}\frac{dy}{d\xi}\right) + 2\right]^2 - \frac{16\xi^2}{xy}\left(\frac{dx}{d\xi}\right)^2} \qquad (1.10)$$

Let us introduce into the examination the function

$$q = \xi\frac{dx}{d\xi} \qquad (1.11)$$

and assume that x is an independent variable. Denoting by dashes differentiation with respect to x, instead of (1.9) and (1.10) we shall have equations

$$y'' = \frac{2\dfrac{y}{q}(x - 3) - 3m\left(y' - \dfrac{y}{x}\right) + (m^2y' - 4)\left[\dfrac{q}{2}\left(\dfrac{y'}{y} + \dfrac{1}{x}\right) - \dfrac{\partial q}{\partial x} - y'\dfrac{\partial q}{\partial y}\right]}{m^2q + (m^2y' - 4)\dfrac{\partial q}{\partial y'}} \qquad (1.12)$$

$$4x - 8 + 2mq\left(\frac{1}{x} - \frac{y'}{y}\right) = \sqrt{\left[mq\left(\frac{1}{x} + \frac{y'}{y}\right) + 2\right]^2 - \frac{16q^2}{xy}} \qquad (1.13)$$

From Eq. (1.13) it is possible to express q in terms of x, y, y' in the form

$$q = 2\,\frac{m\left[4 - \dfrac{9}{x} - \dfrac{y'}{y}(4x - 7)\right] \pm 2\sqrt{m^2\left[\dfrac{y'}{y}(x - 1) + \dfrac{x - 3}{x}\right]^2 - \dfrac{4}{xy}(4x^2 - 16x + 15)}}{-m^2\left(\dfrac{3y'^2}{y^2} - \dfrac{10y'}{xy} + \dfrac{3}{x^2}\right) - \dfrac{16}{xy}} \qquad (1.14)$$

In relationships (1.1–1.7) ϵ is the internal energy per unit volume of the stationary gas, p is the pressure, $\varkappa$ is a constant with dimensionality $T^2M^{-1}L^{-1}$, σ_i are the dimensional constants that enter into additional conditions specifying the problem; the dot denotes differentiation with respect to time coordinate t; and the dash denotes differentiation with respect to space coordinate r.

Let us assume that, among σ_i, on which the unknown functions x and Y depend, there exists a single dimensional

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Thus we arrive at a single ordinary differential equation of the second order (1.12), in which it is necessary to substitute q, $\partial q/\partial x$, $\partial q/\partial y$, and $\partial q/\partial y'$; these quantities are determined with the aid of (1.14). For coordinate velocity and pressure, according to (1.6) and (1.7), we obtain formulas

$$v = \frac{r}{t}V(x) = \frac{r}{t}\,y\left[\frac{2x - 5}{2q} - \frac{m}{4}\left(\frac{3y'}{y} - \frac{1}{x}\right)\right] \qquad (1.15)$$

$$\varkappa p = r^{-2}P(x) = \frac{r^{-2}}{2x}\left[2x - 4 - mq\left(\frac{y'}{y} - \frac{1}{x}\right)\right] \qquad (1.16)$$

where $V(x)$ and $P(x)$ are nondimensional velocity and pressure.

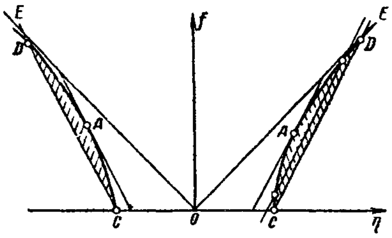

Fig. 1

2. Let us examine conditions at a shock wave as boundary conditions. Here we assume that the motion of a gas in front of and behind the wave is described by Eqs. (1.12) and (1.13). The requirement that there be continuity of energy flow, momentum of matter, and gravitational field through the surface of a shock wave leads to the relationships[1]

$$[x] = [Y] = [\dot{x}D + Y'] = 0 \qquad D = \frac{dr^*(t)}{dt} \tag{2.1}$$

where the quantities in brackets denote the discontinuity in these quantities, while $r = r^*(t)$ is the law of motion of a shock wave (see also Refs. 3–7). The calorimetric equation of state of the ultra-relativistic gas (1.2) completes the system of equations (1.1). However, in order to characterize completely the processes in this gas, it is necessary to introduce other thermodynamic functions, such as temperature, entropy, etc., using the corresponding equations for their determination. It is possible to introduce the density ρ into this examination by determining it from the equation, which is equivalent to the condition of the conservation of entropy of a particle

$$(\rho u^k);_k = 0$$

where the semicolon denotes the covariant derivative. Then, at the surface of a shock wave we should have the condition

$$[\rho u^0(D - v)] = 0 \qquad u^0 = (Y - xv^2)^{-1/2} \tag{2.2}$$

Using the expression for the entropy of the ultra-relativistic gas[8]

$$S = 3 \ln p/\rho^{4/3} \tag{2.3}$$

with the aid of (2.2) we obtain formula

$$\exp\left[\frac{1}{3}(S_1 - S_2)\right] = \frac{p_1}{p_2}\left\{\frac{v_1 - D}{v_2 - D}\sqrt{\frac{Y - xv_2^2}{Y - xv_1^2}}\right\}^{4/3} \tag{2.4}$$

where the indices 1 and 2 are related to the magnitudes on the opposite sides of the discontinuity. In order to confine ourselves within the framework of Eq. (1.2), without introducing new unknowns, let us examine Eq. (1.5) as a condition, which, together with (2.1), links the first derivatives of the gravitational potentials on the opposite sides of a shock wave. For self-similar motions at the surface of a shock wave

$$\xi = 1 \qquad D = -\frac{2}{m}\frac{r}{t}$$

Substitution of (1.8) and (1.11) into (2.1) gives

$$(m^2y_1' - 4)q_1 = (m^2y_2' - 4)q_2 = B \tag{2.5}$$

where B is a continuous function. After eliminating y_i' with the aid of relationship

$$y_i' = \frac{B}{q_i m^2} + \frac{4}{m^2} \qquad (i = 1,2) \tag{2.6}$$

and instead of q_i introducing a new variable

$$\eta_i = q_i + \frac{\left(\dfrac{m}{x} + \dfrac{4}{my}\right)\left(2 + \dfrac{B}{my}\right)}{\left(\dfrac{m}{x} - \dfrac{4}{my}\right)^2} \tag{2.7}$$

the second condition at a shock wave due to Eqs. (1.13) will take on the form

$$f = 2a\eta_i + c = +\sqrt{a^2(\eta_i^2 - b^2)} \tag{2.8}$$

where

$$a = \frac{m}{x} - \frac{4}{my}$$

$$b = \frac{4\left(2 + \dfrac{B}{my}\right)}{\sqrt{xy}\left(\dfrac{m}{x} - \dfrac{4}{my}\right)^2} \tag{2.9}$$

$$c = -\frac{4m\left(2 + \dfrac{B}{my}\right)}{x\left(\dfrac{m}{x} - \dfrac{4}{my}\right)} + 4(x - 1)$$

Equation (2.8) in the plane $[\eta, f]$ determines the points of intersection of a straight line, having a slope $2a$, with the branches of a hyperbola lying in the semiplane $f \geqslant 0$, and having asymptotes $f = \pm a\eta$. The regions of possible double intersections are shown by shaded areas in Fig. 1. Let us assume

$$\eta_i|_{f=0} = -\frac{c}{2a} = nb \tag{2.10}$$

where $\frac{1}{2}\sqrt{3} \leqslant n \leqslant 1$. The points C and D correspond to the value $n = 1$, and point A, in the vicinity of which the jumps y' and q become infinitesimally small, corresponds to the value $n = \frac{1}{2}\sqrt{3}$. On the basis of (2.6–2.10) and (1.15) and (1.16) we obtain

$$q_i = \frac{(x - 1)\left[\pm\dfrac{4}{3\sqrt{xy}}\left(4n \pm \sqrt{4n^2 - 3}\right) - \dfrac{m}{x} - \dfrac{4}{my}\right]}{\left(\dfrac{m}{x} \mp \dfrac{2n}{\sqrt{xy}}\right)\left(\dfrac{m}{x} - \dfrac{4}{my}\right)} \tag{2.11}$$

$$y_i' = \frac{\left[(x - 1)\left(\dfrac{m}{x} - \dfrac{4}{my}\right) - 2\left(\dfrac{m}{x} \mp \dfrac{2n}{\sqrt{xy}}\right)\right] y\left(\dfrac{m}{x} - \dfrac{4}{my}\right)}{m(x - 1)\left[\pm\dfrac{4\left(4n \pm \sqrt{4n^2 - 3}\right)}{3\sqrt{xy}} - \dfrac{m}{x} - \dfrac{4}{my}\right]} + \frac{4}{m^2} \tag{2.12}$$

$$P_i = \pm\frac{2(x - 1)\left(n \pm \sqrt{4n^2 - 3}\right)}{3\sqrt{xy}\left(m \pm 2n\sqrt{x/y}\right)} \tag{2.13}$$

$$V_i = \pm\sqrt{\frac{y}{x}}\left(2n \mp \sqrt{4n^2 - 3}\right) \times \frac{\pm m\sqrt{y/x} - 2\left(2n \pm \sqrt{4n^2 - 3}\right)}{\pm 3m\sqrt{y/x} - 2\left(2n \mp \sqrt{4n^2 - 3}\right)} \tag{2.14}$$

In formulas (2.11–2.14) the upper sign in front of $\sqrt{y/x}$, $\sqrt{xy}$ corresponds to the case when $ab > 0$, whereas the lower sign corresponds to the case when $ab < 0$; the upper sign in front of $\sqrt{4n^2 - 3}$ corresponds to the values after the jump (section AD), whereas the lower sign corresponds to the values before the jump (section AC). In particular, at point

C, where the velocity of the particle reaches the velocity of light

$$V_c = \pm\sqrt{\frac{y}{x}} \qquad P_c = 0$$

and at the corresponding point D of the particle after the jump

$$V_D = \pm\sqrt{\frac{y}{x}}\,\frac{\pm m\sqrt{y/x} - 6}{\pm 3m\sqrt{y/x} - 2}$$

$$P_D = \frac{4(x-1)}{3x(\pm m\sqrt{y/x} - 2)}$$

At point A, where the velocity of the particle reaches the velocity of sound

$$V_A = \pm\sqrt{\frac{y}{3x}}\,\frac{\pm m\sqrt{\frac{3y}{x}} - 6}{\pm m\sqrt{\frac{3y}{x}} - 2}$$

$$P_A = \frac{x-1}{\pm x\sqrt{3xy}\left(\frac{m}{x} \mp \sqrt{\frac{3}{xy}}\right)}$$

When $n > 1$, formulas (2.11) and (2.12) represent only one solution of Eqs. (2.5) and (2.8), which corresponds to the upper sign in front of $\sqrt{4n^2 - 3}$, and in this case the discontinuities are impossible. Since the nondimensional velocity of a shock wave, which is equal to $-2/m$, must be smaller than the velocity of light, then

$$\frac{y}{x} > \frac{4}{m^2} \qquad (2.15)$$

and, hence, am is always greater than zero. Let us formulate the possible cases by taking into account that $P \geqslant 0$ in physical cases:

1) $a < 0,\ b < 0,\ m < 0,\ x < 1$
expansion ($V > 0$), expanding shock wave

2) $a > 0,\ b < 0,\ m > 0,\ x < 1$
convergence ($V < 0$), converging shock wave

For cases 1 and 2 the velocities determined by formula (2.11), when $\frac{1}{2}\sqrt{3} \leqslant n \leqslant 1$, will be greater than the velocity of motion of the jump which is equal to $-2/m$, and will not exceed the velocity of light, which is equal to $\pm\sqrt{y/x}$.

3) $a > 0,\ b > 0,\ m > 0,\ x > 1$
(converging shock wave)

4) $a < 0,\ b > 0,\ m < 0,\ x > 1$
(expanding shock wave)

In addition, for cases 3 and 4 we have

a) If $\sqrt{\frac{y}{x}} > \frac{2}{|m|}(2n + \sqrt{4n^2 - 3})$

then

$$V_{1,2} > 0 \quad \text{for} \quad 3$$
$$V_{1,2} < 0 \quad \text{for} \quad 4$$

b) If $\sqrt{\frac{y}{x}} = \frac{2}{|m|}(2n + \sqrt{4n^2 - 3})$

then

$$V_1 = 0 \qquad V_2 = \frac{2}{m}\sqrt{4n^2 - 3}\,(2n + \sqrt{4n^2 - 3})$$

$$P_1 = \frac{x-1}{3x} \qquad P_2 = \frac{x-1}{3x}\,\frac{n - \sqrt{4n^2-3}}{n + \sqrt{4n^2-3}} \qquad (2.16)$$

c) If $\frac{2}{|m|}(2n + \sqrt{4n^2 - 3}) > \sqrt{\frac{y}{x}} > \frac{2}{|m|}(2n - \sqrt{4n^2 - 3})$

then

$$V_1 V_2 < 0$$

d) If $\sqrt{\frac{y}{x}} = \frac{2}{|m|}(2n - \sqrt{4n^2 - 3})$

then

$$V_1 = -\frac{2}{m}\sqrt{4n^2 - 3}\,(2n - \sqrt{4n^2 - 3}) \qquad V_2 = 0$$

$$(2.17)$$

$$P_1 = \frac{x-1}{3x}\,\frac{n + \sqrt{4n^2-3}}{n - \sqrt{4n^2-3}} \qquad P_2 = \frac{x-1}{3x}$$

e) If $\frac{2}{|m|}(2n - \sqrt{4n^2 - 3}) > \sqrt{\frac{y}{x}} > \frac{2}{|m|}$

then

$$V_{1,2} < 0 \quad \text{for} \quad 3$$
$$V_{1,2} > 0 \quad \text{for} \quad 4$$

Let us examine the case when $m = -1$, which corresponds to the dimensionality σ, equal to LT^{-2}. It is easy to see that within the framework of self-similar motions $m = -1$ is a singular case, when the region occupied by the moving gas may be connected with the stationary one via a shock wave. The equilibrium is written by the following formulas

$$x = \frac{7}{4} \qquad Y = b\sigma r \qquad \varkappa p = \frac{1}{7r^2} \qquad \varkappa\epsilon = \frac{3}{7r^2} \qquad (2.18)$$

where b is a nondimensional constant.

If the diverging wave propagates along a stationary gas that is in front of it (2.18), then, according to formulas (2.11) and (2.12), which are applicable to the case (2.17), we find

$$q_1 = -\frac{21\sqrt{4n^2-3}\,(2n - \sqrt{4n^2-3})}{16(n - \sqrt{4n^2-3})^2} \qquad q_2 = 0$$

$$(2.19)$$

$$y_1' = \frac{16(n - \sqrt{4n^2-3})^2(\sqrt{4n^2-3} - 2n)}{3\sqrt{4n^2-3}} + 4$$

$$y_2' = \infty$$

at the same time

$$D = 2\frac{r}{t} > V_1\frac{r}{t} > 0 \qquad P_1 > P_2 = \frac{1}{7} \qquad V_2 = 0 \qquad (2.20)$$

which corresponds to the transition of the particle through the shock wave from region 2 to region 1. For the entropy, formula (2.4) gives

$$\exp\left[\frac{1}{3}(S_1 - S_2)\right] =$$

$$\frac{n + \sqrt{4n^2-3}}{n - \sqrt{4n^2-3}}\left(\frac{2n^2 - n\sqrt{4n^2-3} - 1}{\sqrt{1-n^2}}\right)^{4/3} \geqslant 1 \qquad (2.21)$$

Analogous to the case when the gas moving toward the center is discontinuously transformed into a stationary state, formulas (2.16), we have

$$q_2 = \frac{21\sqrt{4n^2-3}\,(2n+\sqrt{4n^2-3})}{16\,(n+\sqrt{4n^2-3})^2} \qquad q_1 = 0$$

$$y_2' = \frac{16(n+\sqrt{4n^2-3})^2\,(2n+\sqrt{4n^2-3})}{3\sqrt{4n^2-3}} + 4 \qquad y_1' = \infty \tag{2.22}$$

Here again the particle passes from region 2 into region 1 and

$$\exp\left[\frac{1}{3}(S_1 - S_2)\right] = \frac{n+\sqrt{4n^2-3}}{n-\sqrt{4n^2-3}} \times \left(\frac{\sqrt{1-n^2}}{2n^2+n\sqrt{4n^2-3}-1}\right)^{4/3} \geqslant 1 \tag{2.23}$$

Let us note that when $n \to 1$, the right-hand sides of formulas (2.21) and (2.23) tend toward ∞. This means that the jump, during which the velocity of the particle changes from the velocity of light to zero velocity, is accompanied by an infinite jump in entropy.

3. Let us examine now, when $m = -1$, the asymptotic solutions of the systems (1.9) and (1.10) in the region of values $t = 0, r = \infty$ and $t = \infty, r = 0$.

Let us examine the case when the $x(\xi)$ and $y(\xi)$, in the region $\xi^\alpha = 0$, are expanded into series of the form

$$x = \xi^s(a_0 + a_1\xi^\alpha + a_2\xi^{2\alpha} + \ldots)$$

$$y = \xi^\tau(b_0 + b_1\xi^\alpha + b_2\xi^{2\alpha} + \ldots) \tag{3.1}$$

where $\alpha, s, \tau, a_i, b_i$ are constants. Substituting (3.1) in Eqs. (1.9) and (1.10), it is easy to show that the solution of the (3.1) type may exist only in the case when $s = 0$, $\tau = ka$, where k is a whole number not exceeding 2. Let us examine thoroughly the possibilities existing here.

1) Case $k = 2$. Substituting series (3.1) into relationships (1.15) and (1.16), we shall obtain

$$V = \frac{b_0\xi^\alpha}{2\alpha a_1}\left\{2a_0 - 5 + 3\alpha + \xi^\alpha\left[2a_1 + \frac{3\alpha b_1}{2b_0} - \frac{a_1\alpha}{2a_0} + (2a_0 - 5 + 3\alpha)\left(\frac{b_1}{b_0} - \frac{2a_2}{a_1}\right)\right] + \ldots\right\} \tag{3.2}$$

$$P = \frac{1}{2a_0}\left[2a_0 - 4 + 2\alpha + \xi^\alpha\left(\frac{\alpha b_1}{b_0} - \frac{3\alpha a_1}{a_0} + \frac{4a_1}{a_0}\right) + \ldots\right] \tag{3.3}$$

In addition, we have

$$\sqrt{\frac{y}{x}} = \sqrt{\frac{b_0}{a_0}}\,\xi^\alpha\left[1 + \frac{\xi^\alpha}{2}\left(\frac{b_1}{b_0} - \frac{a_1}{a_0}\right) + \ldots\right] \tag{3.4}$$

If $P > 0$ when $\xi^\alpha = 0$, then

$$\alpha > 2 - a_0 \tag{3.5}$$

In this case, with the aid of Eqs. (1.9) and (1.10), it is possible to express successively a_i, b_i in terms of a_0, b_0, and α. For the first coefficients we obtain

$$a_1^2 = \frac{a_0 b_0}{4\alpha^2}\left[(1-\alpha)^2 - (2a_0 - 4 + 2\alpha)^2\right] \tag{3.6}$$

$$b_1 = \frac{a_1 b_0(8a_0^2 + 4\alpha a_0 - 20a_0 - 3\alpha - \alpha^2 + 12)}{a_0\alpha(7 - 4a_0 - 3\alpha)}$$

$$a_2 = \frac{a_1 b_1}{4b_0} - \frac{b_0}{4\alpha^2}(a_0 - 3) + \frac{b_0}{4} - \frac{3b_0}{4\alpha} + \frac{a_1^2}{4a_0} \tag{3.7}$$

On the basis of (3.5) and (3.6) we conclude that either $a_0 < 1$, $\alpha > 1$, or $a_0 > 1$, $\alpha < 1$. Then from (3.6) it follows that

$$(1-\alpha)^2 > (2a_0 - 4 + 2\alpha)^2 + (2a_0 - 5 + 3\alpha)^2$$

Whence, on the basis of (3.5):

$$0 > (\alpha - 1) + (2a_0 - 4 + 2\alpha)$$

Since $2a_0 - 4 + 2\alpha > 0$, and if P does not transform into zero, the only possibilities are that $a_0 > 1$, $\alpha < 1$. Then from (3.5) and (3.6) we obtain

$$2 - \alpha < a_0 < \tfrac{1}{2}(5 - 3\alpha) \tag{3.8}$$

In this case $|V| < \sqrt{y/x}$ for small ξ^α. If $a_0 = 2 - \alpha$, then $V = \sqrt{y/x}$, $P = 0$ when $\xi^\alpha \to 0$. When $\alpha = 1$, then $a_0 = 1$, $a_i = b_i = 0$, and formulas

$$x = 1 \qquad y = b_0\xi^2 \qquad \xi = \sigma t^2/r \tag{3.9}$$

represent an exact solution of the system (1.9 and 1.10), corresponding to the Galilean metric.

If $0 < \alpha < 1$, then ξ^α is small when $t \to 0$, $r \to \infty$. When $r \to \infty$ the dimensional pressure $\varkappa p = r^{-2}P$ tends toward zero, whereas the velocity of the particle, which in the laboratory system of coordinates is equal to $v\sqrt{x}$, tends toward infinity together with the velocity of light. The velocity of the particle measured in terms of its own time for a given point in space is equal to

$$v_c = c\sqrt{\frac{x}{y}}\,V$$

where c is the velocity of light in a vacuum; in this instance it remains less than c. Since, according to (3.8):

$$2a_0 - 5 + 3\alpha < 0 \tag{3.10}$$

then $V < 0$ if $a_1 > 0$, which corresponds to an increase in x from the value a_0 for an increase in ξ^α, and $V > 0$, if $a_1 < 0$. In the last case x decreases. When $t = 0$:

$$\varkappa p = \frac{1}{a_0}(a_0 - 2 + \alpha)r^{-2}$$

$$\frac{u_c}{c} = \frac{\sqrt{a_0 b_0}}{2\alpha a_1}(2a_0 - 5 + 3\alpha) \tag{3.11}$$

whereas v and $\sqrt{y/x}$ tend to zero when $\alpha > \frac{1}{2}$, and to infinity when $\alpha < \frac{1}{2}$. If $\alpha < 0$, then ξ^α is small, when $r \to 0$, $t \to \infty$. If $r \to 0$, when $\alpha > 2 - a_0$, then $p \to \infty$, and u_c takes on the value given by (3.11).

When $t \to \infty$, the functions p and u_c tend toward values given by (3.11), whereas v and $\sqrt{y/x}$ tend toward zero.

2) Case when $k = 1$. Then

$$V = \frac{b_0}{2\alpha a_1}\left\{2a_0 - 5 + \frac{3}{2}\alpha + \xi^\alpha\left[2a_1 + \frac{3\alpha b_1}{2b_0} - \frac{\alpha a_1}{2a_0} + \left(2a_0 - 5 + \frac{3}{2}\alpha\right)\left(\frac{b_1}{b_0} - \frac{2a_2}{a_1}\right)\right] + \ldots\right\} \tag{3.12}$$

$$\sqrt{\frac{y}{x}} = \sqrt{\frac{b_0}{a_0}}\,\xi^{\alpha/2}\left[1 + \frac{\xi^\alpha}{2}\left(\frac{b_1}{b_0} - \frac{a_1}{a_0}\right) + \ldots\right]$$

$$P = \frac{1}{2a_0}\left\{2a_0 - 4 + \alpha + \xi^\alpha\left[2a_1 + \frac{\alpha b_1}{b_0} - \frac{\alpha a_1}{a_0} - \frac{a_1}{a_0}(2a_0 - 4 + \alpha)\right] + \ldots\right\}$$

The requirement that $V < \sqrt{y/x}$ leads to the condition

$$a_0 = \tfrac{5}{2} - \tfrac{3}{4}\alpha$$

Since $P = (2 - \alpha)/4a_0 > 0$ when $\xi^\alpha = 0$, then $\alpha < 2$.

Table 1

x	y	q	dx/dy	ξ	$-V$	P
1.7500	36.7054	0.7622	0.01917	1.0000	2.2437	0.04215
1.7052	34.3054	0.7065	0.01816	0.9409	2.3257	0.03815
1.6455	30.9054	0.6311	0.01700	0.8604	2.4246	0.03302
1.6056	28.5054	0.5802	0.01634	0.8054	2.4756	0.02977
1.5545	25.3054	0.5152	0.01563	0.7336	2.5192	0.02582
1.5053	22.1054	0.4532	0.01510	0.6626	2.5322	0.02224
1.4576	18.9054	0.3937	0.01477	0.5918	2.5113	0.01897
1.4106	15.7054	0.3362	0.01464	0.5201	2.4489	0.01594
1.3636	12.5054	0.2801	0.01479	0.4463	2.3308	0.01315
1.3155	9.3054	0.2242	0.01537	0.3682	2.1425	0.01062
1.2642	6.1054	0.1671	0.01685	0.2825	1.8597	0.00797
1.2362	4.5054	0.1369	0.01836	0.2347	1.6600	0.00660
1.2131	3.3054	1.1127	0.02029	0.1892	1.4640	0.00560
1.1868	2.1054	0.08585	0.02385	0.1449	1.2070	0.00452
1.1716	1.5054	0.07062	0.02718	0.1192	1.0399	0.00392
1.1601	1.1054	0.05925	0.03082	0.1017	0.9035	0.00349
1.1502	0.8054	0.04966	0.03523	0.0849	0.7805	0.00313
1.1386	0.5054	0.03849	0.04320	0.0574	0.6265	0.00275
1.1276	0.2804	0.02808	0.05641	0.0442	0.4725	0.00239
1.1231	0.2054	0.02382	0.06516	0.0371	0.4066	0.00224
1.1196	0.1554	0.02058	0.07426	0.0317	0.3547	0.00216
1.1035	0.0687	0.02140	0.11196	0.0200	0.2473	0.00185
1.0971	0.0196	0.00654	0.21169	0.0100	0.1304	0.00167
1.0900	0.0000	0.00000	$+\infty$	0.0000	0.0000	0.00140

With the aid of Eqs. (1.9) and (1.10) we find

$$a_1 = \frac{b_0}{4\alpha^2}(\alpha - 2)(\alpha - 1)$$

$$b_1 = \frac{b_0^2(\alpha - 2)(\alpha - 1)(9\alpha^2 - 21\alpha + 22)}{\alpha^3(2\alpha - 5)(10 - 3\alpha)}$$

Then, according to (3.12):

$$V = \frac{2b_0(4\alpha^3 - 29\alpha^2 + 46\alpha - 22)}{\alpha^2(5 - 2\alpha)(10 - 3\alpha)}\xi^\alpha + \dots$$

The coefficient $a_1 < 0$, when $1 < \alpha < 2$, and, hence, x decreases from the value a_0 in the interval $\frac{7}{4} > a_0 > 1$ when ξ^α increases. When $\alpha = 1$, we obtain solution (2.18). If $\alpha < 1$, then $a_1 > 0$; the function x increases from $a_0 > \frac{7}{4}$ when ξ^α increases. In all cases $V < 0$. The value of $\alpha = 2$ corresponds to solution (3.9). When $\xi^\alpha \to 0$, then $u_c \to 0$, in contrast to the case when $k = 2$.

If $k = 0$, then solutions of the (3.1) type do not exist. When $k < 0$, we obtain solutions (2.18) and (3.9) and

$$x = \tfrac{3}{4} \qquad y = b_0\xi^3$$

which has no physical meaning.

4. As an example, let us examine the case when the ultra-relativistic gas moves toward a central point. This process schematically represents the formation of a central nebula. Let the boundary of the stationary central nucleus represented by a shock wave, inside which the state of the gas is described by formulas (2.18), expand according to the law

$$r = \sigma t^2 \tag{4.1}$$

The quantity n in formulas (2.16) and (2.22), which determine the magnitudes of the jumps of functions characterizing the motion, depends on the initial values of the functions cited. Let us assume arbitrarily that $n = 0.9$. Then on the external side of a shock wave we shall have

$$x = 1.75 \qquad y = 36.7054 \qquad y' = 52.1587 \qquad q = 0.7622 \tag{4.2}$$

$$P = 0.04215 \qquad V = -2.2436 \qquad \sqrt{\frac{y}{x}} = 4.5798 \tag{4.3}$$

Formula (2.23) for the ratio of the entropy of a particle before and after the passage of the jump gives $\exp[\frac{1}{3}(S_1 - S_2)] = 1.03503$.

From Eq. (1.11) we shall find

$$\xi = \exp\int_{7/4}^{x}\frac{d\lambda}{q(\lambda)} \tag{4.4}$$

The calculation shows that it is necessary to take the minus sign in front of the root for the value of q in formula (1.14). Expressions in (4.2) represent the Cauchy conditions with respect to Eq. (1.18). The results of the numerical solution of this problem are given in Table 1. Fig. 2 depicts the graphs of the functions $y = y(x)$ and $\sigma t^2/r = \xi(x)$, which were determined according to (4.4); the same figure compares $V(x)$ with the functions $\sqrt{y/x}$, which represent the maximum allowable values of $V(x)$. The same figure also depicts the graph of the particle velocity u_c/c, which is measured in terms of the particle time for each point in space [see (3.10)]. Fig. 3 depicts the graph of the nondimensional pressure $P = P(x)$.

As r approaches infinity for a fixed t, the functions x, y, P, $\sqrt{y/x}$ decrease monotonically together with ξ. The function $V(x)$, on attaining a certain maximum, begins to decrease,

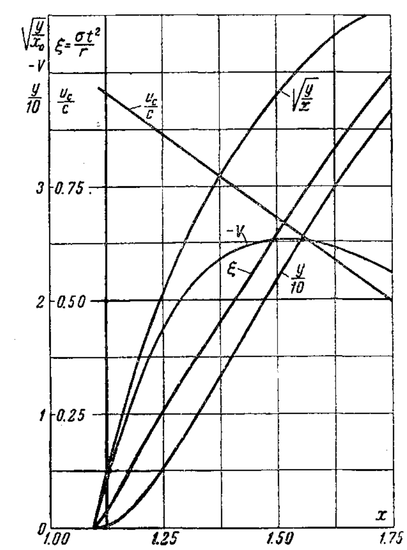

Fig. 2

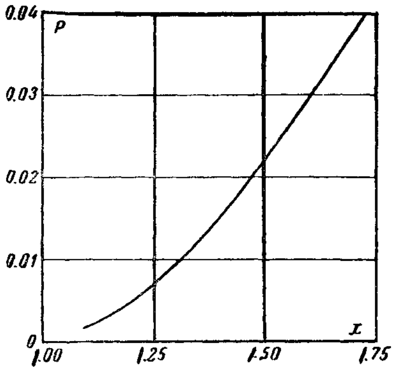

Fig. 3

approaching the expression $\sqrt{y/x}$, and is equal to the value of this function only when it transforms into zero. At the same time, the value u_c increases monotonically; however, it is always less than the velocity of light in a vacuum c. The point $\xi = 0$, $y = 0$, $x = a_0$ is a singular point of the system (1.9 and 1.10). Let us use the asymptotic representations of the unknown functions of the form (3.1–3.3) in the vicinity of this point by selecting approximately

$$\alpha = 0.9116 \qquad a_0 = 1.09 \qquad b_0 = 87.4225 \tag{4.5}$$

With the aid of (3.6) and (3.7) we obtain

$$\begin{aligned} x &+ 1.09 + 04730\xi^{\alpha} + 0.1474\xi^{2\alpha} + \ldots \\ y &= 87.4225\xi^{2\alpha} - 49.8508\xi^{3\alpha} + \ldots \\ V &= -8.6371\xi^{\alpha} - 2.7385\xi^{2\alpha} + \ldots \\ P &= 0.00147 + 0.0134\xi^{\alpha} + \ldots \end{aligned} \tag{4.6}$$

The time scale in the system of coordinates (1.3) may be selected arbitrarily. If we assume

$$t^* = \frac{1}{2\alpha h} t^{2\alpha}$$

where h is a constant with dimensionality $T^{2\alpha-1}$, then when $t^* = 0$ and $r > 0$ we obtain the following initial values for the functions that characterize the field:

$$\begin{aligned} Y^* = Y\left(\frac{dt}{dt^*}\right)^2 &= b_0 h^2 \sigma^{2\alpha} r^{2(1-a)} \qquad x = a_0 \\ \frac{dr}{dt^*} &= -V_0 h \sigma^{\alpha} r^{1-\alpha} \qquad \varkappa d = P_0 r^{-2} \end{aligned} \tag{4.7}$$

where V_0, P_0 are the corresponding constants.

If we employ for each point in space its own time

$$\tau = \frac{1}{c}\int_0^t \sqrt{Y}\, dt$$

then the initial values of pressure and velocity are represented by formulas (3.11).

Thus, if at the initial moment of time the particles of the medium have a distribution of pressure, velocity, and gravitational potential close to the distribution given by formula (4.7) or (3.11), then all further motion of the gas will conform to the solution derived here.

It is pointed out that the paper represents an attempt to obtain an exact solution for a nonstationary motion of a continuous medium taking into account shock waves within the formulation of the general theory of relativity. The difficult nature of the problem and the lack of preliminary results make it necessary to limit the initial study to self-similar motions. Despite certain artificiality of conditions, general nature of motions can be discovered by such a technique.

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